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Nonlinear analysis of members curved in space with warping and Wagner effects

Yong-Lin Pi [a,*], Mark Andrew Bradford [a], Brian Uy [b]

[a] *School of Civil and Environmental Engineering, University of New South Wales, Sydney, NSW 2052, Australia*
[b] *School of Civil, Mining and Environmental Engineering, University of Wollongong, Sydney, NSW 2522, Australia*

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Abstract

The centroidal axis of a member that is curved in space is generally a space curve. The curvature of the space curve is not necessarily in the direction of either of the principal axes of the cross-section, but can be resolved into components in the directions of both of these principal axes. Hence, a member curved in space is primarily subjected to combined compressive, biaxial bending and torsional actions under vertical (or gravity) loading. In addition, warping actions in particular may occur in curved members with an open thin-walled cross-section, and as the deformations increase, significant interactions of the compressive, biaxial bending and torsional actions occur and profoundly nonlinear deformations are developed in the nonlinear range of structural response. This makes the nonlinear behaviour of a member curved in space very complicated, making it difficult to obtain a consistent differential equation of equilibrium for the nonlinear analysis of members curved in space. In addition, because torsion is one of the primary actions in these members, when the torsional deformations become large, the Wagner effects including both Wagner moment and the conjugate Wagner strain terms are increasingly significant and need to be included in the nonlinear analysis. This paper takes advantage of the merits of so-called "geometrically exact beam theory" and the weak form formulation of the differential equations of equilibrium in beam theory, and it develops consistent differential equations of equilibrium for the nonlinear elastic analysis of members curved in space with warping and Wagner effects. The application of the nonlinear differential equations of equilibrium to various problems is illustrated.

Keywords: Analysis; Differentiation; Equilibrium; Member curved in space; Nonlinear; Wagner effects; Warping

* Corresponding author. Tel.: +61 293855066; fax: +61 293856039.
E-mail addresses: y.pi@unsw.edu.au (Y.-L. Pi), m.bradford@unsw.edu.au (M.A. Bradford), b.uy@unsw.edu.au (B. Uy).

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1. Introduction

Curved members are used very widely in engineering structures (Fig. 1), yet their generic analysis is little understood. When a member is curved in elevation (an arch) or is curved in plan, its centroidal axis is a plane curve and its curvature is usually in the direction of one of the principal axes of the cross-section. However, when a member is more arbitrarily curved in space, its centroidal axis is a space curve rather than a plane curve. The curvature of the space curve is not necessarily in the direction of either principal axis of the cross-section, but it can be resolved into components in the directions of both principal axes. Hence, the member curved in space is primarily subjected to combined compressive, biaxial bending and torsional actions under vertical loading. In addition, warping actions in particular may occur in curved members with an open thin-walled cross-section. The nonlinear behaviour of a member curved in space is even more complicated because significant interactions of these compressive, biaxial bending and torsional actions are developed in the nonlinear range of structural response, and because of this complication studies of the nonlinear analysis of members that are curved in space appear to be very limited in the open literature and concentrate on the theoretical aspects. Reissner (1981) reported a study on the finite deformation of members curved in space. Crisfield and Jelenić (1999) discussed the objectivity of strain measures based on geometrically exact three-dimensional theory, while Atluri et al. (2001) developed a consistent theory of finite stretches and rotations in members curved in space. However, nonlinear differential equations of equilibrium for members curved in space including the effects of warping and Wagner effects (Wagner, 1936) have rarely been reported. Yoda et al. (1978, 1980) and Hirashima et al. (1979) developed a finite displacement theory of naturally curved and twisted thin-walled members and took warping and Wagner strain terms into account in their strain expression. It is known that the Wagner effects include both the Wagner terms in the finite strains and the corresponding Wagner moment (Pi and Trahair, 1995; Pi and Bradford, 2000). To account for the Wagner effects properly, both the Wagner terms in the finite strains and the corresponding Wagner moment need to be considered. However, the significance of the Wagner strain terms and the Wagner moment in the large twist torsional analysis has not widely been recognised, although the theoretical study of Yoda et al. (1978) included both the Wagner strain terms and the Wagner moment in their formulation. As pointed out by Iura and Hirashima (1985), in the early studies of naturally curved and twisted members, the second order terms were neglected in the constitutive equations for twisting moment. Pi and Trahair (1995) developed a nonlinear torsional theory for straight members and found

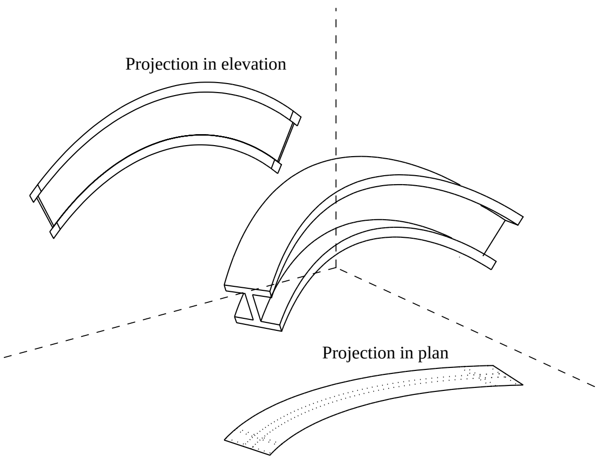

Fig. 1. Member curved in space and its projections in plan and in elevation.

that the Wagner terms in the finite strains and the corresponding Wagner moment play an important part in the large twist rotation analysis of beams subjected to pure torsional action, and that in order to consider the Wagner effects correctly, both the Wagner terms in the finite strains and the corresponding Wagner moment have to be included in the nonlinear analysis as shown in Fig. 2. It can be seen from Fig. 2 that without Wagner moment and Wagner strain terms, quite soft large deformation behaviour is predicted (Farwell and Galambos, 1969; Pi and Trahair, 1995). Without Wagner terms in the finite strain or without the corresponding Wagner moment, the Wagner effects cannot be accounted for correctly. It is also worth pointing out that without the Wagner moment, even the flexural–torsional buckling load of circular arches under uniform compression cannot be predicted correctly (Pi and Bradford, 2002).

So-called ‘‘geometrically exact beam theory’’ derived directly from the resultant forms of the differential equations of equilibrium has been used extensively for the geometric nonlinear analysis of beams (Reissner, 1973, 1981; Simo and Vu-Quoc, 1986; Crisfield, 1990; Ibrahimbegivić et al., 1995; Jelenić et al., 1995; Crisfield and Jelenić, 1999; Atluri et al., 2001). The incorporation of the shear strains that are needed to model ‘‘Timoshenko beams’’ is easy with this method and it is also convenient for the use of various mathematical tools such as vector and tensor analysis, differential geometry, and geometric (Clifford) algebra (McRobie and Lasenby, 1999) in the formulation. The differential equations of equilibrium are written in terms of stress resultants, and constitutive models are usually expressed in terms of relationships between the stress resultants and three ‘‘direct’’ strains and three curvatures. Because it is difficult to include the warping strain and its corresponding stress resultant in geometrically exact beam theory, most of these studies have not considered warping in detail except that of Simo and Vu-Quoc (1991) who treated the warping deformation by including the warping amplitude in their derivation of the relationship between the bimoment and the warping strain. In addition, the Wagner term within the finite strains and its corresponding stress resultant-Wagner moment (Wagner, 1936) do not appear to have been included in the models based on ‘‘geometrically exact beam theory’’. The first Piola–Kirchhoff stress vector used in the ‘‘geometrically exact beam theory’’ may imply that the Wagner moment is included in the stress resultants. However, it is known that the strain measures corresponding to the Wagner moment is the second order twist which has not been included in the existing ‘‘geometrically exact beam theory’’. Because of this, the constitutive relationship

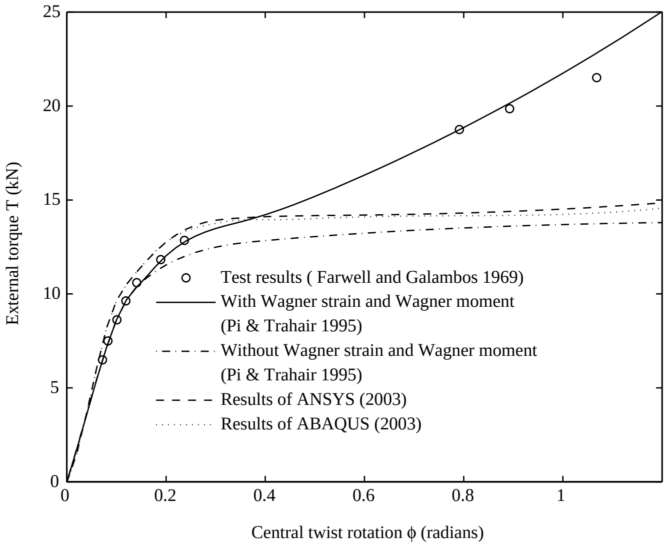

Fig. 2. Effects of Wagner strain terms and Wagner moment.

between the stress resultant Wagner moment and the conjugate strain measure do not appear to have been developed in the existing ''geometrically exact beam theory'' as well. Hence, it is difficult to use the existing ''geometrically exact beam theory'' to predict large torsional responses. For example, the beam elements of the commercial finite element packages ABAQUS (2003) and ANSYS (2003) that are related to the ''geometrically exact beam theory'' cannot correctly predict the large torsional response test results (Farwell and Galambos, 1969) as shown in Fig. 2. Furthermore, the extension of the elastic model based on ''geometrically exact beam theory'' to elasto-plastic analysis have more obstacles because it is difficult to establish an accurate, unified and effective constitutive model that relates the strain measures or the deformations to the stress resultants.

The weak form of the continuum equilibrium equations that are based on the virtual work principle in terms of stresses and virtual strains is also often used for nonlinear analysis of beams (Bathe and Bolourchi, 1979; Kitipornchai and Chan, 1990; Dvorkin et al., 1988; Bild et al., 1992; Pi and Trahair, 1994; Ronagh and Bradford, 1999; Chan and Gu, 2000). Compared with ''geometrically exact beam theory'', this method has two advantages. Firstly, it is easy to incorporate some important terms in the expression for the finite strains such as warping and Wagner terms and the corresponding stress resultants in the formulation that do not appear in the models that are based on ''geometrically exact beam theory''. Secondly, because constitutive models based on this method are usually expressed in the form of more routine stress–strain relationships, it is much easier to extend these models from a nonlinear elastic analysis to a nonlinear elasto-plastic analysis. Four independent parameters in the formulation, viz. the three displacements of the member axis and twist rotation of the cross-section, are often used in these models to describe the deformations and strains. One of the drawbacks of formulations that use these four parameters is that approximations often need to be made by introducing restrictions on the magnitudes of the displacements and limiting the rotation to be small, so that the formulations can be facilitated readily. However, approximations or simplifications that are made in the earlier stages of the derivation may not be able to separate significant rigid body motion from the real deformations and thus may produce over-stiff solution due to self straining (Simo and Vu-Quoc, 1987; Kane et al., 1987; Crisfield, 1990; Banerjee and Lemak, 1991; Pi and Trahair, 1994; Pi and Bradford, 2002).

The purpose of this paper is to take advantage of the merits of ''geometric exact beam theory'' and the formulation of the continuum equilibrium equations in weak form in the development of a consistent theory and of differential equations of equilibrium for the nonlinear analysis of members curved in space with warping and Wagner effects.

2. Axis systems and kinematics

A body attached (material) right-handed curvilinear orthogonal axis system is used here to describe the motion of a member curved in space during its deformation. In the undeformed configuration, the curvilinear orthogonal axis system is in the position *oxys*. The axis *os* passes through the centroids of the cross-section of the undeformed curved member and the axes *ox* and *oy* coincide with the principal axes of the cross-section, as shown in Fig. 3. It is worth pointing out that the centroidal axis *os* is generally a space curve rather than a plane curve. A unit vector $\mathbf{p}_s$ in the tangent direction of the axis *os*, and unit vectors $\mathbf{p}_x$ and $\mathbf{p}_y$ in the direction of the axes *ox* and *oy* form a right-handed orthonormal basis. The unit vectors $\mathbf{p}_x$, $\mathbf{p}_y$, $\mathbf{p}_s$ are used as the fixed reference basis. They do not change with the deformation, but their directions change from point to point along the member axis *os*.

After the deformation, the origin *o* displaces u, v, w to o^* and the cross-sections (that are assumed to remain rigid in their planes and so do not distort) rotate through an angle ϕ, and so the body attached curvilinear orthogonal axis system moves and rotates to a new position $o^*x^*y^*s^*$ as shown in Fig. 3. In the deformed configuration, a unit vector $\mathbf{q}_s$ is defined along the tangent direction of the axis o^*s^* of the axis

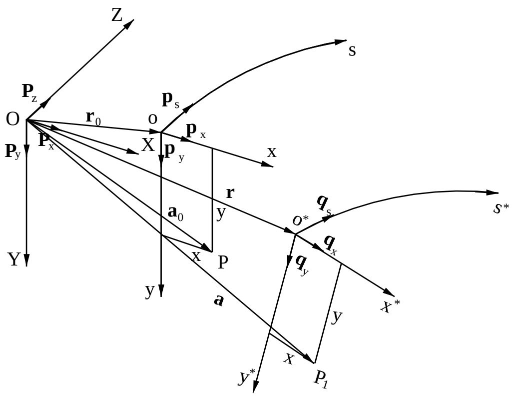

Fig. 3. Axes system, basis and position vectors.

system $o^*x^*y^*s^*$, and unit vectors $\mathbf{q}_x$ and $\mathbf{q}_y$ are defined along the principal axes o^*x^*, o^*y^* of the rotated cross-section at o^*. The unit vectors $\mathbf{q}_x$, $\mathbf{q}_y$, $\mathbf{q}_s$ also form a orthonormal basis. They attach to the member and move with the member during the deformation with the vector $\mathbf{q}_s$ normal to the cross-section at all times.

A fixed (space) right-handed rectangular coordinate system $OXYZ$ is defined in space as also shown in Fig. 3. The position of the undeformed and deformed member can be defined in the axis system $OXYZ$, with unit vectors $\mathbf{P}_X$, $\mathbf{P}_Y$, $\mathbf{P}_Z$ in directions OX, OY and OZ forming a right-handed orthogonal basis.

Before the deformation, the position vector of the centroid o in the fixed axes $OXYZ$ is $\mathbf{r}_0$ (Fig. 3), and so the unit vector $\mathbf{p}_s$ tangential to the centroidal axis os can be expressed in terms of the position vector $\mathbf{r}_0$ as (Pi et al., 2003)

$$\mathbf{p}_s = \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s} \tag{1}$$

In general, the centroidal axis os of a member curved in space has an initial curvature κ_0 and an initial twist κ_{s0}, and the curvature κ_0 is not necessarily in the direction of either principal axis of the cross-section. In this case, the curvature κ_0 can be resolved into components κ_{x0} and κ_{y0} about the unit vectors $\mathbf{p}_x$ and $\mathbf{p}_y$ (i.e. about the axes ox and oy). The Frenet–Serret formulae in terms of basis vectors $\mathbf{p}_x$, $\mathbf{p}_y$, $\mathbf{p}_s$ for members curved in space in the undeformed configuration can then be written as (Pi et al., 2003)

$$\left[\frac{\mathrm{d}\mathbf{p}_x}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{p}_y}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{p}_s}{\mathrm{d}s}\right] = [\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_s]\mathbf{K}_0 \tag{2}$$

where the skew-symmetric matrix $\mathbf{K}_0$ for the initial curvatures and twist is given by

$$\mathbf{K}_0 = \begin{bmatrix} 0 & -\kappa_{s0} & \kappa_{y0} \\ \kappa_{s0} & 0 & -\kappa_{x0} \\ -\kappa_{y0} & \kappa_{x0} & 0 \end{bmatrix} \tag{3}$$

In the deformed configuration, the position vector of the centroid o^* in the fixed axis system $OXYZ$ is $\mathbf{r}$ as shown in Fig. 3, and so the vector $\mathbf{q}_s$ tangential to the deformed centroidal axis o^*s^* can be expressed in terms of the position vector $\mathbf{r}$ of the centroid o^* as

$$\mathbf{q}_s = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s^*} = \frac{1}{1+\epsilon}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \tag{4}$$

where $\mathrm{d}s^* = (1+\epsilon)\mathrm{d}s$ is used, with ϵ being the longitudinal normal strain at the centroid. Because the differentiation of the position vector is taken with respect to the deformed length s^*, $\mathbf{q}_s$ is a unit vector.

The position vector $\mathbf{r}$ of the centroid o^* can be expressed as (Fig. 3)

$$\mathbf{r} = \mathbf{r}_0 + u\mathbf{p}_x + v\mathbf{p}_y + w\mathbf{p}_s \tag{5}$$

Hence, Eqs. (2)–(5) can be used to obtain

$$\mathbf{q}_s = \frac{1}{1+\epsilon}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{1}{1+\epsilon}[\tilde{u}'\mathbf{p}_x + \tilde{v}'\mathbf{p}_y + (1+\tilde{w}')\mathbf{p}_s] = \hat{u}'\mathbf{p}_x + \hat{v}'\mathbf{p}_y + \hat{w}'\mathbf{p}_s \tag{6}$$

where $\tilde{u}' \equiv u' + w\kappa_{y0} - v\kappa_{s0}$, $\tilde{v}' \equiv v' - w\kappa_{x0} + u\kappa_{s0}$, $\tilde{w}' \equiv w' - u\kappa_{y0} + v\kappa_{x0}$, $()' \equiv \mathrm{d}()/\mathrm{d}s$, $\hat{u}' \equiv \tilde{u}'/(1+\epsilon)$, $\hat{v}' \equiv \tilde{v}'/(1+\epsilon)$, $\hat{w}' \equiv (1+\tilde{w}')/(1+\epsilon)$. Because $\mathbf{q}_s$ is a unit vector, it follows from Eq. (6) that

$$(\hat{u}')^2 + (\hat{v}')^2 + (\hat{w}')^2 = 1. \tag{7}$$

from which only two of $\hat{u}'$, $\hat{v}'$ and $\hat{w}'$ are independent.

In the deformed configuration, and in accordance with the Frenet–Serret formulae, the relationship between the derivatives of the basis vectors and the curvatures and twist can be written as (Pi et al., 2003)

$$\left[\frac{\mathrm{d}\mathbf{q}_x}{\mathrm{d}s^*}, \frac{\mathrm{d}\mathbf{q}_y}{\mathrm{d}s^*}, \frac{\mathrm{d}\mathbf{q}_s}{\mathrm{d}s^*}\right] = \frac{1}{1+\epsilon}\left[\frac{\mathrm{d}\mathbf{q}_x}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{q}_y}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{q}_s}{\mathrm{d}s}\right] = [\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_s]\mathbf{K} \tag{8}$$

where the skew-symmetric matrix $\mathbf{K}$ for the curvatures and twist after the deformation is given by

$$\mathbf{K} = \begin{bmatrix} 0 & -\kappa_s & \kappa_y \\ \kappa_s & 0 & -\kappa_x \\ -\kappa_y & \kappa_x & 0 \end{bmatrix} \tag{9}$$

where κ_x and κ_y are the curvatures about the unit vectors $\mathbf{q}_x$ and $\mathbf{q}_y$, i.e. about the axes o^*x^* and o^*y^* respectively, and κ_s is the twist about the unit vector $\mathbf{q}_s$ i.e. about the o^*s^* axis after deformation.

3. Rotations, curvatures and deformations

A rotation matrix $\mathbf{R}$ is often used to describe the relationship between the basis vectors $\mathbf{p}_x$, $\mathbf{p}_y$, $\mathbf{p}_s$ in the undeformed configuration and the basis vectors $\mathbf{q}_x$, $\mathbf{q}_y$, $\mathbf{q}_s$ in the deformed configuration. The rotation matrix for the special orthogonal rotation group $SO(3)$ can be obtained as (Pi et al., 2003)

$$\mathbf{q}_i = \mathbf{R}\mathbf{p}_i, \qquad i = x, y, s \tag{10}$$

or collectively as

$$[\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_s] = [\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_s]\mathbf{R} \tag{11}$$

where the matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{bmatrix} R_{xx} & R_{xy} & R_{xs} \\ R_{yx} & R_{yy} & R_{ys} \\ R_{sx} & R_{sy} & R_{ss} \end{bmatrix} \tag{12}$$

with

$$R_{xx} = (1 - \lambda\hat{u}'^2)C - \lambda\hat{u}'\hat{v}'S, \quad R_{xy} = -(1 - \lambda\hat{u}'^2)S - \lambda\hat{u}'\hat{v}'C, \quad R_{xs} = \hat{u}' \tag{13}$$

$$R_{yx} = (1 - \lambda\hat{v}'^2)S - \lambda\hat{u}'\hat{v}'C, \quad R_{yy} = (1 - \lambda\hat{v}'^2)C + \lambda\hat{u}'\hat{v}'S, \quad R_{ys} = \hat{v}' \tag{14}$$

$$R_{sx} = -\hat{u}'C - \hat{v}'S, \qquad R_{sy} = \hat{u}'S - \hat{v}'C, \qquad R_{ss} = \hat{w}' \tag{15}$$

and where $C \equiv \cos\phi$, $S \equiv \sin\phi$ and $\lambda \equiv 1/(1 + \hat{w}')$.

The components of the rotation matrix $\mathbf{R}$ in Eqs. (13) and (15) are expressed by $\hat{u}$, $\hat{v}$, $\hat{w}$ and ϕ. It is noted that the parameters $\hat{u}$, $\hat{v}$ and $\hat{w}$ need to satisfy the condition given by Eq. (7). Hence, only two of them are independent and the components of the rotation matrix $\mathbf{R}$ given in Eqs. (13) and (15) are expressed in fact by three independent parameters. The components in Eqs. (13) and (15) become infinite only when $1 + \hat{w}' = 0$ and/or $1 + \epsilon = 0$, which do not occur for a real structure.

The rotation matrix given by Eq. (12) satisfies the orthogonality condition (Burn, 2001)

$$\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I} \tag{16}$$

and the unimodular condition that

$$\det \mathbf{R} = +1 \tag{17}$$

where $\det\mathbf{R}$ is the determinant of the matrix $\mathbf{R}$. Hence, the matrix $\mathbf{R}$ belongs to the special orthogonal Lie group for three dimensional rotation $SO(3)$, for which the invariant requirement needed for the rigid body rotation is satisfied.

Differentiating Eq. (11) with respect to s produces

$$\left[\frac{\mathrm{d}\mathbf{q}_x}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{q}_y}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{q}_s}{\mathrm{d}s}\right] = \left[\frac{\mathrm{d}\mathbf{p}_x}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{p}_y}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{p}_s}{\mathrm{d}s}\right]\mathbf{R} + \left[\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_s\right]\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} \tag{18}$$

and substituting Eqs. (2), (11), (18) into Eq. (8), and using $\mathrm{d}s^* = (1 + \epsilon)\,\mathrm{d}s$ yields

$$(1 + \epsilon)\mathbf{K} = \mathbf{R}^{\mathrm{T}}\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} + \mathbf{R}^{\mathrm{T}}\mathbf{K}_0\mathbf{R} \tag{19}$$

It is worth pointing out that the second term $\mathbf{R}^{\mathrm{T}}\mathbf{K}_0 R$ of Eq. (19) embodies the effects of the initial curvatures and twist on the deformed curvatures and twist and leads to some significant terms in the finite strains caused by interactions of the initial curvatures and twist with the finite rotation and extension. However, this term appears to be simplified as $\mathbf{K}_0$ by some researchers.

Substituting Eqs. (12), (13), (15) into Eq. (19) leads to the curvatures κ_x and κ_y and the twist (Love, 1927) κ_s after the deformation as

$$\begin{aligned}\kappa_x = \big\{&\hat{u}''S - \hat{v}''C - \lambda\hat{w}''(\hat{u}'S - \hat{v}'C) + [\lambda(1 - \hat{u}'^2 - \hat{w}'^2)C - \lambda\hat{u}'\hat{v}'S + \hat{w}'C]\kappa_{x0} + [\lambda(1 - \hat{v}'^2 - \hat{w}'^2)S \\ &- \lambda\hat{u}'\hat{v}'C + \hat{w}'S]\kappa_{y0} - (\hat{v}'S + \hat{u}'C)\kappa_{s0}\big\}(1 + \epsilon)^{-1}\end{aligned} \tag{20}$$

$$\begin{aligned}\kappa_y = \big\{&\hat{u}''C + \hat{v}''S - \lambda\hat{w}''(\hat{u}'C + \hat{v}'S) - [\lambda(1 - \hat{u}'^2 - \hat{w}'^2)S + \lambda\hat{u}'\hat{v}'C + \hat{w}'S]\kappa_{x0} + [\lambda(1 - \hat{v}'^2 - \hat{w}'^2)C \\ &+ \lambda\hat{u}'\hat{v}'S + \hat{w}'C]\kappa_{y0} - (\hat{v}'C - \hat{u}'S)\kappa_{s0}\big\}(1 + \epsilon)^{-1}\end{aligned} \tag{21}$$

$$\kappa_s = [\phi' + \lambda(\hat{u}''\hat{v}' - \hat{u}'\hat{v}'') + \hat{u}'\kappa_{x0} + \hat{v}'\kappa_{y0} + \hat{w}'\kappa_{s0}](1 + \epsilon)^{-1} \tag{22}$$

The position vector $\mathbf{a}_0$ of an arbitrary point $P(x, y)$ on the cross-section of the curved member in the undeformed configuration can be expressed as (Fig. 3)

$$\mathbf{a}_0 = \mathbf{r}_0 + x\mathbf{p}_x + y\mathbf{p}_y \tag{23}$$

where $\mathbf{r}_0$ is the position vector of the centroid o in the fixed axes $OXYZ$.

In the undeformed configuration, the initial gradient tensor $\mathbf{F}_0$ can be expressed as

$$\mathbf{F}_0 = \left[\frac{\partial \mathbf{a}_0}{\partial x}, \frac{\partial \mathbf{a}_0}{\partial y}, \frac{\partial \mathbf{a}_0}{\partial s}\right] \tag{24}$$

The position of the point $P(x, y)$ in the deformed configuration is determined based on the following two assumptions. Firstly, it is assumed that curved members are considered to satisfy the Bernoulli hypothesis i.e. the cross-sectional plane remains plane and perpendicular to the member axis during the deformation, while secondly, the total deformation of a point P is assumed to result from two successive motions: a translation and a finite rotation of the cross-section, and a superimposed warping displacement along the unit vector $\mathbf{q}_s$ in the deformed configuration. The magnitude of this warping displacement is assumed to be given by the product of two functions: the normalized section warping function $\omega(x, y)$ and the change of twist $(\kappa_s - \kappa_{s0})$. Under these two assumptions, the position vector $\mathbf{a}$ of the point P_1, which is the position of the point P after the deformation, can be expressed in terms of the basis vectors $\mathbf{q}_x$, $\mathbf{q}_y$, $\mathbf{q}_s$ in Fig. 3 as

$$\mathbf{a} = \mathbf{r} + x\mathbf{q}_x + y\mathbf{q}_y - \omega(x, y)(\kappa_s - \kappa_{s0})\mathbf{q}_s \tag{25}$$

in which $\mathbf{r}$ is the position vector of the centroid o^* after the deformation in the fixed axis system $OXYZ$ and is given by Eq. (5).

In the deformed configuration, the deformation gradient tensor $\mathbf{F}$ can be expressed as

$$\mathbf{F} = \left[\frac{\partial \mathbf{a}}{\partial x}, \frac{\partial \mathbf{a}}{\partial y}, \frac{\partial \mathbf{a}}{\partial s}\right] = \left[\frac{\partial \mathbf{a}}{\partial x}, \frac{\partial \mathbf{a}}{\partial y}, (1+\epsilon)\frac{\partial \mathbf{a}}{\partial s^*}\right] \tag{26}$$

since $\mathrm{d}s^* = (1+\epsilon)\mathrm{d}s$.

4. Finite strains

The strain tensor can be expressed in terms of the initial and deformation gradient tensors as

$$\begin{bmatrix} \epsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xs} \\ \frac{1}{2}\gamma_{yx} & \epsilon_{yy} & \frac{1}{2}\gamma_{ys} \\ \frac{1}{2}\gamma_{sx} & \frac{1}{2}\gamma_{sy} & \epsilon_{ss} \end{bmatrix} = \frac{1}{2}\left\{\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{F}_0^{\mathrm{T}}\mathbf{F}_0\right\} \tag{27}$$

The components of the strain tensor given in Eq. (27) can then obtained by substituting Eqs. (23) and (25) into Eq. (27) as

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{xy} = \epsilon_{yx} = 0 \tag{28}$$

$$\begin{aligned}\epsilon_{ss} &= \frac{1}{2}\left[\frac{\partial \mathbf{a}}{\partial s}\cdot\frac{\partial \mathbf{a}}{\partial s} - \frac{\partial \mathbf{a}_0}{\partial s}\cdot\frac{\partial \mathbf{a}_0}{\partial s}\right] \\ &= \frac{1}{2}[(1+\epsilon)^2 - 1] - x[\kappa_y(1+\epsilon) - \kappa_{y0}] + y(\kappa_x(1+\epsilon) - \kappa_{x0}) - \omega(\kappa_s - \kappa_{s0})'(1+\epsilon) + \frac{1}{2}x^2[\kappa_y^2(1+\epsilon)^2 \\ &\quad - \kappa_{y0}^2] + \frac{1}{2}y^2[\kappa_x^2(1+\epsilon)^2 - \kappa_{x0}^2] + \frac{1}{2}\omega^2[\kappa_s - \kappa_{s0}]^2(1+\epsilon)^2 - xy[\kappa_x\kappa_y(1+\epsilon)^2 - \kappa_{x0}\kappa_{y0}] \\ &\quad + x\omega\kappa_y(\kappa_s - \kappa_{s0})(1+\epsilon)^2 - y\omega\kappa_x(\kappa_s - \kappa_{s0})(1+\epsilon)^2 + \frac{1}{2}(x^2 + y^2)[\kappa_s^2(1+\epsilon)^2 - \kappa_{s0}^2]\end{aligned} \tag{29}$$

$$\gamma_{sy} = \gamma_{ys} = \left[\frac{\partial \mathbf{a}}{\partial y}\cdot\frac{\partial \mathbf{a}}{\partial s} - \frac{\partial \mathbf{a}_0}{\partial y}\cdot\frac{\partial \mathbf{a}_0}{\partial s}\right] = \left(x - \frac{\partial \omega(x, y)}{\partial y}\right)(\kappa_s - \kappa_{s0}) \tag{30}$$

and

$$\gamma_{sx} = \gamma_{xs} = \left[\frac{\partial \mathbf{a}}{\partial x} \cdot \frac{\partial \mathbf{a}}{\partial s} - \frac{\partial \mathbf{a}_0}{\partial x} \cdot \frac{\partial \mathbf{a}_0}{\partial s}\right] = -\left(y + \frac{\partial \omega(x,y)}{\partial x}\right)(\kappa_s - \kappa_{s0}) \tag{31}$$

Substituting Eqs. (20)–(22) into Eq. (29), ignoring the effects of the third and higher order terms, the longitudinal normal strain ϵ_{ss} at the displaced point P_1 can be expressed as

$$\begin{aligned}\epsilon_{ss} = {} & \tilde{w}' + \frac{1}{2}\tilde{u}'^2 + \frac{1}{2}\tilde{v}'^2 + \frac{1}{2}\tilde{w}'^2 \\ & - x\left\{\tilde{u}''C + \tilde{v}''S - \kappa_{x0}S - \frac{1}{2}\kappa_{x0}\tilde{w}'S + \kappa_{y0}\left[\left(1 - \frac{1}{2}\tilde{v}'^2 + \frac{1}{2}\tilde{w}'\right)C - 1\right] - (\tilde{v}'C - \tilde{u}'S)\kappa_{s0}\right\} \\ & + y\left\{\tilde{u}''S - \tilde{v}''C + \kappa_{y0}S + \frac{1}{2}\kappa_{y0}\tilde{w}'S + \kappa_{x0}\left[\left(1 - \frac{1}{2}\tilde{u}'^2 + \frac{1}{2}w'\right)C - 1\right] - (\tilde{v}'S + \tilde{u}'C)\kappa_{s0}\right\} \\ & - \omega\left\{\phi'' + \tilde{u}''\kappa_{x0} + \tilde{v}''\kappa_{y0} + \tilde{w}''\kappa_{s0} + \frac{1}{2}(\tilde{u}'''\tilde{v}' - \tilde{u}'\tilde{v}''')\right\} + \frac{1}{2}(x^2 + y^2)\{\phi' + \tilde{u}'\kappa_{x0} + \tilde{v}'\kappa_{y0} + \tilde{w}'\kappa_{s0}\}^2\end{aligned} \tag{32}$$

where the approximation $\lambda = 1/(1 + \hat{w}) = 1/(2 + \tilde{w}') \approx 1/2$ has been used.

Substituting Eq. (22) into Eqs. (31) and (30) leads to the shear strains γ_{sy} and γ_{sx} as

$$\gamma_{sy} = \left(x - \frac{\partial \omega(x,y)}{\partial y}\right)\left\{\phi' + \tilde{u}'\kappa_{x0} + \tilde{v}'\kappa_{y0} + \tilde{w}'\kappa_{s0} + \frac{1}{2}(\tilde{u}''\tilde{v}' - \tilde{u}'\tilde{v}'')\right\} \tag{33}$$

$$\gamma_{sx} = -\left(y + \frac{\partial \omega(x,y)}{\partial x}\right)\left\{\phi' + \tilde{u}'\kappa_{x0} + \tilde{v}'\kappa_{y0} + \tilde{w}'\kappa_{s0} + \frac{1}{2}(\tilde{u}''\tilde{v}' - \tilde{u}'\tilde{v}'')\right\} \tag{34}$$

The warping function $\omega(x,y)$ can be obtained by considering the Saint–Venant uniform torsion problem for a prismatic bar, which results in the Laplace equation

$$\nabla^2\omega = \frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = 0 \tag{35}$$

and which may be solved by considering that the shear stresses τ_{sx} and τ_{sy} conjugate to the shear strains γ_{sy} and γ_{sx} satisfy a traction-free boundary condition on the lateral surface. The solution for the warping function $\omega(x,y)$ can be uniquely specified by using the following three additional orthogonality conditions

$$\int_A \omega(x,y)\,\mathrm{d}A = \int_A x\omega(x,y)\,\mathrm{d}A = \int_A y\omega(x,y)\,\mathrm{d}A = 0 \tag{36}$$

For thin-walled sections, additional assumptions can be made to simplify the solutions. For example, Vlasov's hypothesis (Vlasov, 1961), that the shear deformations in the mid-line of a thin-walled plate are extremely small and can be neglected, can be used to obtain the warping function from Eq. (35) for a doubly symmetric I-section as

$$\omega(x,y) = \begin{cases} x(y+h) & \text{for the top flange} \\ -xy & \text{for the web} \\ x(y-h) & \text{for the bottom flange} \end{cases} \tag{37}$$

where h is the distance between the flange centroids.

5. Stresses and stress resultants

The stress vector $\boldsymbol{\sigma}$ at the point P_1 can be written as

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\epsilon} \tag{38}$$

where the stress vector is given by

$$\boldsymbol{\sigma} = \{\sigma_{ss}, \tau_{sy}, \tau_{sx}\}^{\mathrm{T}} \tag{39}$$

with σ_{ss} being the longitudinal normal stress, and where τ_{sy} and τ_{sx} are the shear stresses due to uniform torsion; the elastic modulus matrix $\mathbf{E}$ is given by

$$\mathbf{E} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \tag{40}$$

with E and G being the Young's and shear moduli of elasticity respectively, and the strain vector $\boldsymbol{\epsilon}$ is given by

$$\boldsymbol{\epsilon} = \{\epsilon_{ss}, \gamma_{sy}, \gamma_{sx}\}^{\mathrm{T}} \tag{41}$$

with ϵ_{ss}, γ_{sy} and γ_{sx} being given by Eqs. (32)–(34), respectively.

The vector of stress resultants referred to the deformed axes $o^*x^*y^*s^*$ acting on the cross-section can be written as

$$\boldsymbol{\Sigma} = \{R_1, R_2, R_3, R_4, R_5, R_6\}^{\mathrm{T}} = \{N, -M_y, M_x, B, W, T\}^{\mathrm{T}} = \int_A \mathbf{C}^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}A \tag{42}$$

where $\mathbf{C}$ is a 3×6 matrix and given by

$$\mathbf{C} = \begin{bmatrix} 1 & x & y & \omega & x^2+y^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(x - \frac{\partial\omega(x,y)}{\partial y}\right) \\ 0 & 0 & 0 & 0 & 0 & -\left(y + \frac{\partial\omega(x,y)}{\partial x}\right) \end{bmatrix} \tag{43}$$

and where N is the axial compression, M_y is the bending moment about the axis o^*y^*, M_x is the bending moment about the axis o^*x^*, B is the bimoment, W is the Wagner moment (Wagner, 1936), and T is the torque due to uniform torsion, whence

$$N = \int_A \sigma_{ss}\,\mathrm{d}A \tag{44}$$

$$M_y = -\int_A \sigma_{ss}x\,\mathrm{d}A \tag{45}$$

$$M_x = \int_A \sigma_{ss}y\,\mathrm{d}A \tag{46}$$

$$B = \int_A \sigma_{ss}\omega\,\mathrm{d}A \tag{47}$$

$$W = \int_A \sigma_{ss}(x^2+y^2)\,\mathrm{d}A \tag{48}$$

and

$$T=\int_A\left[\left(x-\frac{\partial\omega(x,y)}{\partial y}\right)\tau_{sy}-\left(y+\frac{\partial\omega(x,y)}{\partial x}\right)\tau_{sx}\right]\mathrm{d}A \tag{49}$$

6. Nonlinear equilibrium equations

The nonlinear equilibrium equations for a member curved in space can be derived from the principle of virtual work which requires that

$$\mathrm{d}U=\int_V \mathrm{d}\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}V-\int_0^S(\mathrm{d}\mathbf{u}_q^{\mathrm{T}}\mathbf{q}+\mathrm{d}\boldsymbol{\theta}_m^{\mathrm{T}}\mathbf{m}_{\mathrm{e}})\,\mathrm{d}s-\sum_{k=1,2}(\mathrm{d}\mathbf{u}_{Q_k}^{\mathrm{T}}\mathbf{Q}_k+\mathrm{d}\boldsymbol{\theta}_{M_{\mathrm{e}k}}^{\mathrm{T}}\mathbf{M}_{\mathrm{e}k})=0 \tag{50}$$

for all admissible sets of infinitesimal virtual displacements $\{\mathrm{d}u,\mathrm{d}v,\mathrm{d}w,\mathrm{d}\phi\}$ and the corresponding virtual strains $\mathrm{d}\boldsymbol{\epsilon}$ given by

$$\mathrm{d}\boldsymbol{\epsilon}=\{\mathrm{d}\epsilon_{ss},\mathrm{d}\gamma_{sy},\mathrm{d}\gamma_{sx}\} \tag{51}$$

where $\mathbf{q}$ and $\mathbf{Q}_k(k=1,2)$ are the external distributed and concentrated loads at both ends of the member and are given by

$$\mathbf{q}=\{q_x,q_y,q_s\}^{\mathrm{T}}\quad\text{and}\quad\mathbf{Q}_k=\{Q_{xk},Q_{yk},Q_{sk}\}^{\mathrm{T}}\quad(k=1,2) \tag{52}$$

with q_x, q_y and q_s being the distributed loads in the direction of the axes ox, oy and os while Q_{xk}, Q_{yk} and Q_{sk} are the concentrated loads in the direction of the axes ox, oy and os; and $\mathbf{m}_{\mathrm{e}}$ and $\mathbf{M}_{\mathrm{e}k}(k=1,2)$ are the external distributed and concentrated moments at both ends of the member and are given by

$$\mathbf{m}_{\mathrm{e}}=\{m_{\mathrm{e}x},m_{\mathrm{e}y},m_{\mathrm{e}s}\}^{\mathrm{T}}\quad\text{and}\quad\mathbf{M}_{\mathrm{e}k}=\{M_{\mathrm{e}xk},M_{\mathrm{e}yk},M_{\mathrm{e}sk}\}^{\mathrm{T}}\quad(k=1,2) \tag{53}$$

with $m_{\mathrm{e}x}$, $m_{\mathrm{e}y}$ and $m_{\mathrm{e}s}$ being the distributed moments about the axes ox, oy and os while $M_{\mathrm{e}xk}$, $M_{\mathrm{e}yk}$ and $M_{\mathrm{e}sk}$ are the concentrated moments about the axes ox, oy and os.

The virtual longitudinal normal strain $\mathrm{d}\epsilon_{ss}$ can be obtained by taking the variation of Eq. (32) as

$$\begin{aligned}
\mathrm{d}\epsilon_{ss}=&\left\{\left[\tilde{v}'\kappa_{s0}-(1+\tilde{w}')\kappa_{y0}\right]\mathrm{d}u+\tilde{u}'\,\mathrm{d}u'+[(1+\tilde{w}')\kappa_{x0}-\tilde{u}'\kappa_{s0}]\,\mathrm{d}v\right.\\
&\left.+\tilde{v}'\,\mathrm{d}v'+(\tilde{u}'\kappa_{y0}-\tilde{v}'\kappa_{x0})\,\mathrm{d}w+(1+\tilde{w}')\,\mathrm{d}w'\right\}\\
&-x\left\{\left[\frac{1}{2}(S\kappa_{x0}-C\kappa_{y0})\kappa_{x0}-C(\kappa_{y0}\tilde{v}'+\kappa_{s0})\kappa_{s0}\right]\mathrm{d}u+2S\kappa_{s0}\,\mathrm{d}u'+C\,\mathrm{d}u''\right.\\
&+\left[\frac{1}{2}(C\kappa_{y0}-S\kappa_{x0})\kappa_{y0}-S\kappa_{s0}^2\right]\mathrm{d}v-C(\tilde{v}'\kappa_{y0}+2\kappa_{s0})\,\mathrm{d}v'+S\,\mathrm{d}v''+\left[(C\kappa_{x0}+S\kappa_{y0})\kappa_{s0}+C\tilde{v}'\kappa_{x0}\kappa_{y0}\right]\mathrm{d}w\\
&\left.+\frac{1}{2}(C\kappa_{y0}-S\kappa_{x0})\,\mathrm{d}w'+\left[C\tilde{v}''-C\kappa_{x0}\left(1+\frac{1}{2}\tilde{w}'\right)-S\tilde{u}''-S\kappa_{y0}\left(1+\frac{1}{2}\tilde{w}'\right)-(S\tilde{v}'+C\tilde{u}')\kappa_{s0}\right]\mathrm{d}\phi\right\}\\
&+y\left\{-\left[\frac{1}{2}(C\kappa_{y0}+S\kappa_{x0})\kappa_{y0}+S\kappa_{s0}^2\right]\mathrm{d}u-C(\tilde{u}'\kappa_{x0}+2\kappa_{s0})\,\mathrm{d}u'+S\,\mathrm{d}u''\right.\\
&+\left[\frac{1}{2}(C\kappa_{x0}+S\kappa_{y0})\kappa_{x0}+C(\tilde{u}'\kappa_{x0}+\kappa_{s0})\kappa_{s0}\right]\mathrm{d}v-2S\kappa_{s0}\,\mathrm{d}v'-C\,\mathrm{d}v''\\
&+\left[(S\kappa_{x0}-C\kappa_{y0})\kappa_{s0}-C\tilde{u}'\kappa_{x0}\kappa_{y0}\right]\mathrm{d}w+\frac{1}{2}(C\kappa_{x0}+S\kappa_{y0})\,\mathrm{d}w'
\end{aligned}$$

$$+\left[S\tilde{v}''-S\kappa_{x0}\left(1+\frac{1}{2}\tilde{w}'\right)+C\tilde{u}''+C\kappa_{y0}\left(1+\frac{1}{2}\tilde{w}'\right)-(C\tilde{v}'-S\tilde{u}')\kappa_{s0}\right]\mathrm{d}\phi\Big\}$$
$$-\omega\Big\{\frac{1}{2}\tilde{u}'''\kappa_{s0}\,\mathrm{d}u-\frac{1}{2}\tilde{v}'''\,\mathrm{d}u'+\left(\kappa_{x0}-\frac{1}{2}\tilde{u}'\kappa_{s0}\right)\mathrm{d}u''+\frac{1}{2}\tilde{v}'\,\mathrm{d}u'''+\frac{1}{2}\tilde{v}'''\kappa_{s0}\,\mathrm{d}v+\frac{1}{2}\tilde{u}'''\,\mathrm{d}v'$$
$$+\left(\kappa_{y0}-\frac{1}{2}\tilde{v}'\kappa_{s0}\right)\mathrm{d}v''-\frac{1}{2}\tilde{u}'\,\mathrm{d}v'''-\frac{1}{2}(\tilde{u}'''\kappa_{x0}+\tilde{v}'''\kappa_{y0})\,\mathrm{d}w+\left[\kappa_{s0}+\frac{1}{2}(\tilde{v}'\kappa_{y0}+\tilde{u}'\kappa_{x0})\right]\mathrm{d}w''+\mathrm{d}\phi''\Big\}$$
$$+(x^2+y^2)\left[\phi'+\tilde{u}'\kappa_{x0}+\tilde{v}'\kappa_{y0}+\tilde{w}'\kappa_{s0}\right]\left\{\kappa_{x0}\,\mathrm{d}u'+\kappa_{y0}\,\mathrm{d}v'+\kappa_{s0}\,\mathrm{d}w'+\mathrm{d}\phi'\right\} \tag{54}$$

while the virtual uniform torsion shear strains $\mathrm{d}\gamma_{sy}$ and $\mathrm{d}\gamma_{sx}$ can be obtained by taking the variation of Eqs. (33) and (34) as

$$\mathrm{d}\gamma_{sy}=\left(x-\frac{\partial\omega(x,y)}{\partial y}\right)\Big\{\frac{1}{2}\tilde{u}''\kappa_{s0}\mathrm{d}u+\left(\kappa_{x0}-\frac{1}{2}\tilde{v}''-\frac{1}{2}\tilde{u}'\kappa_{s0}\right)\mathrm{d}u'+\frac{1}{2}\tilde{v}'\,\mathrm{d}u''+\frac{1}{2}\tilde{v}''\kappa_{s0}\,\mathrm{d}v$$
$$+\left(\kappa_{y0}+\frac{1}{2}\tilde{u}''-\frac{1}{2}\tilde{v}'\kappa_{s0}\right)\mathrm{d}v'-\frac{1}{2}\tilde{u}'\,\mathrm{d}v''-\frac{1}{2}\left(\tilde{u}''\kappa_{x0}+\tilde{v}''\kappa_{y0}\right)\mathrm{d}w$$
$$+\left[\kappa_{s0}+\frac{1}{2}\left(\tilde{v}'\kappa_{y0}+\tilde{u}'\kappa_{x0}\right)\right]\mathrm{d}w'+\mathrm{d}\phi'\Big\} \tag{55}$$

and

$$\mathrm{d}\gamma_{sx}=-\left(y+\frac{\partial\omega(x,y)}{\partial x}\right)\Big\{\frac{1}{2}\tilde{u}''\kappa_{s0}\mathrm{d}u+\left(\kappa_{x0}-\frac{1}{2}\tilde{v}''-\frac{1}{2}\tilde{u}'\kappa_{s0}\right)\mathrm{d}u'+\frac{1}{2}\tilde{v}'\,\mathrm{d}u''+\frac{1}{2}\tilde{v}''\kappa_{s0}\,\mathrm{d}v$$
$$+\left(\kappa_{y0}+\frac{1}{2}\tilde{u}''-\frac{1}{2}\tilde{v}'\kappa_{s0}\right)\mathrm{d}v'-\frac{1}{2}\tilde{u}'\,\mathrm{d}v''-\frac{1}{2}\left(\tilde{u}''\kappa_{x0}+\tilde{v}''\kappa_{y0}\right)\mathrm{d}w$$
$$+\left[\kappa_{s0}+\frac{1}{2}\left(\tilde{v}'\kappa_{y0}+\tilde{u}'\kappa_{x0}\right)\right]\mathrm{d}w'+\mathrm{d}\phi'\Big\} \tag{56}$$

If the external loads are assumed to be acting at the centroids,

$$\mathrm{d}\mathbf{u}_q=\mathrm{d}\mathbf{u}_Q=\{\mathrm{d}u,\mathrm{d}v,\mathrm{d}w\}^{\mathrm{T}}\quad\text{and}\quad\mathrm{d}\boldsymbol{\theta}_m=\mathrm{d}\boldsymbol{\theta}_M=\{-\mathrm{d}\tilde{v}',\mathrm{d}\tilde{u}',\mathrm{d}\phi\}^{\mathrm{T}} \tag{57}$$

By substituting Eqs. (54)–(57) into Eq. (50) and considering Eq. (42), the virtual work given by Eq. (50) can be expressed in terms of stress resultants, external loads, and displacements and their derivatives. Then, integrating Eq. (50) by parts allows the differential equilibrium equations for members curved in space to be separated as

$$N[\tilde{v}'\kappa_{s0}-(1+\tilde{w}')\kappa_{y0}]-[N\tilde{u}']'-M_x\left[\frac{1}{2}(C\kappa_{y0}+S\kappa_{x0})\kappa_{y0}+S\kappa_{s0}^2\right]$$
$$+[M_xC(\tilde{u}'\kappa_{x0}+2\kappa_{s0})]'+[M_xS]''+M_y\left[\frac{1}{2}(S\kappa_{x0}-C\kappa_{y0})\kappa_{x0}-C(\kappa_{y0}\tilde{v}'+\kappa_{s0})\kappa_{s0}\right]$$
$$-2[M_yS\kappa_{s0}]'+[M_yC]''-\frac{1}{2}B\tilde{u}'''\kappa_{s0}-\frac{1}{2}[B\tilde{v}''']'-\left[B\left(\kappa_{x0}-\frac{1}{2}\tilde{u}'\kappa_{s0}\right)\right]''+\frac{1}{2}[B\tilde{v}']'''-\frac{1}{2}T\tilde{u}''\kappa_{s0}$$
$$+\left[T\left(\kappa_{x0}-\frac{1}{2}\tilde{v}''-\frac{1}{2}\tilde{u}'\kappa_{s0}\right)\right]'-\frac{1}{2}[T\tilde{v}']''-[W(\phi'+\tilde{u}'\kappa_{x0}+\tilde{v}'\kappa_{y0}+\tilde{w}\kappa_{s0})\kappa_{x0}]'=q_x-m'_{ey} \tag{58}$$

for bending about the axis oy,

$$N[(1+\tilde{w}')\kappa_{x0}-\tilde{u}'\kappa_{s0}]-[N\tilde{v}']'+M_x\left[\frac{1}{2}\left(C\kappa_{x0}+S\kappa_{y0}\right)\kappa_{x0}+C(\tilde{u}'\kappa_{x0}+\kappa_{s0})\kappa_{s0}\right]+2[M_xS\kappa_{s0}]'$$
$$-[M_xC]''+M_y\left[\frac{1}{2}\left(C\kappa_{y0}-S\kappa_{x0}\right)\kappa_{y0}-S\kappa_{s0}^2\right]+[M_yC(\tilde{v}'\kappa_{y0}+2\kappa_{s0})]'+[M_yS]''-\frac{1}{2}B\tilde{v}'''\kappa_{s0}$$
$$+\frac{1}{2}[B\tilde{u}''']'-\left[B\left(\kappa_{y0}-\frac{1}{2}\tilde{v}'\kappa_{s0}\right)\right]''-\frac{1}{2}[B\tilde{u}']'''-\frac{1}{2}T\tilde{v}''\kappa_{s0}+\left[T\left(\kappa_{y0}+\frac{1}{2}\tilde{u}''-\frac{1}{2}\tilde{v}'\kappa_{s0}\right)\right]'$$
$$+\frac{1}{2}[T\tilde{u}']''-[W(\phi'+\tilde{u}'\kappa_{x0}+\tilde{v}'\kappa_{y0}+\tilde{w}'\kappa_{s0})\kappa_{y0}]'=q_y+m'_{ex} \tag{59}$$

for bending about the axis ox,

$$N(\tilde{u}'\kappa_{y0}-\tilde{v}'\kappa_{x0})-[N(1+\tilde{w}')]'+M_x[(S\kappa_{x0}-C\kappa_{y0})\kappa_{s0}-C\tilde{u}'\kappa_{x0}\kappa_{y0}]-\frac{1}{2}[M_x(\kappa_{y0}S+\kappa_{x0}C)]'$$
$$+M_y[(C\kappa_{x0}+S\kappa_{y0})\kappa_{s0}+C\tilde{v}'\kappa_{x0}\kappa_{y0}]-\frac{1}{2}[M_y(\kappa_{y0}C-\kappa_{x0}S)]'+\frac{1}{2}B[\tilde{u}'''\kappa_{x0}+\tilde{v}'''\kappa_{y0}]-[B(\kappa_{s0}$$
$$+\frac{1}{2}\tilde{v}'\kappa_{y0}+\frac{1}{2}\tilde{u}'\kappa_{x0})]''+\frac{1}{2}T(\tilde{u}''\kappa_{x0}+\tilde{v}''\kappa_{y0})+\left[T\left(\kappa_{s0}+\frac{1}{2}\tilde{v}'\kappa_{y0}+\frac{1}{2}\tilde{u}'\kappa_{x0}\right)\right]'$$
$$-[W(\phi'+\tilde{u}'\kappa_{x0}+\tilde{v}'\kappa_{y0}+\tilde{w}\kappa_{s0})\kappa_{s0}]'=q_s+m_{ey}\kappa_{y0}+m_{ex}\kappa_{x0} \tag{60}$$

for axial compression in the direction of the axis os, and

$$M_x\left[\tilde{u}''C+\tilde{v}''S+\kappa_{y0}C\left(1+\frac{1}{2}\tilde{w}'\right)-\kappa_{x0}S\left(1+\frac{1}{2}\tilde{w}'\right)-(\tilde{v}'C-\tilde{u}'S)\kappa_{s0}\right]$$
$$-M_y\left[\tilde{u}''S-\tilde{v}''C+\kappa_{y0}S\left(1+\frac{1}{2}\tilde{w}'\right)+\kappa_{x0}C\left(1+\frac{1}{2}\tilde{w}'\right)+(\tilde{v}'S+\tilde{u}'C)\kappa_{s0}\right]-B''+T'$$
$$-\left[W(\phi'+\tilde{u}'\kappa_{x0}+\tilde{v}'\kappa_{y0}+\tilde{w}\kappa_{s0})\right]'=m_{es} \tag{61}$$

for torsion about the axis os.

The boundary conditions can be stated as

$$N\tilde{u}'-M_xC(\tilde{u}'\kappa_{x0}+2\kappa_{s0})-[M_xS]'+2M_yS\kappa_{s0}-[M_yC]'+\frac{1}{2}B\tilde{v}'''+\left[B\left(\kappa_{x0}-\frac{1}{2}\tilde{u}'\kappa_{s0}\right)\right]'$$
$$-\frac{1}{2}[B\tilde{v}']''+\frac{1}{2}T\tilde{v}''-T\left(\kappa_{x0}-\frac{1}{2}\tilde{u}'\kappa_{s0}\right)+\frac{1}{2}[T\tilde{v}']'+W\left(\phi'+\tilde{u}'\kappa_{x0}+\tilde{v}'\kappa_{y0}+\tilde{w}'\kappa_{s0}\right)\kappa_{x0}$$
$$=Q_x+m_{ey}\quad\text{or}\quad \mathrm{d}u=0, \tag{62}$$

$$M_yC+M_xS+\frac{1}{2}[B\tilde{v}']'-B\left(\kappa_{x0}-\frac{1}{2}\tilde{u}'\kappa_{s0}\right)+\frac{1}{2}T\tilde{v}'=M_{ey}\quad\text{or}\quad \mathrm{d}u'=0 \tag{63}$$

and

$$-\frac{1}{2}B\tilde{v}'=0\quad\text{or}\quad \mathrm{d}u''=0 \tag{64}$$

for bending about the axis oy,

$$N\tilde{u}' - 2M_xS\kappa_{s0} + [M_xC]' - M_yC(\tilde{v}'\kappa_{y0} + 2\kappa_{s0}) - [M_yS]' - \frac{1}{2}B\tilde{v}''' + \left[B(\kappa_{y0} - \frac{1}{2}\tilde{u}'\kappa_{s0})\right]' + \frac{1}{2}[B\tilde{u}']''$$
$$-\frac{1}{2}T\tilde{u}'' - T\left(\kappa_{y0} - \frac{1}{2}\tilde{v}'\kappa_{s0}\right) - \frac{1}{2}[T\tilde{u}']' + W(\phi' + \tilde{u}'\kappa_{x0} + \tilde{v}'\kappa_{y0} + \tilde{w}'\kappa_{s0})\kappa_{y0}$$
$$= Q_y - m_{ex} \quad \text{or} \quad \mathrm{d}v = 0, \tag{65}$$

$$M_yS - M_xC - B\left(\kappa_{y0} - \frac{1}{2}\tilde{v}'\kappa_{s0}\right) - \frac{1}{2}[B\tilde{u}']' + \frac{1}{2}T\tilde{u}' = -M_{ex} \quad \text{or} \quad \mathrm{d}v' = 0 \tag{66}$$

and

$$\frac{1}{2}B\tilde{u}' = 0 \quad \text{or} \quad \mathrm{d}v'' = 0 \tag{67}$$

for bending about the axis ox,

$$N + \frac{1}{2}M_x[\kappa_{y0}S + \kappa_{x0}C] + \frac{1}{2}M_y[\kappa_{y0}C - \kappa_{x0}S] + \left\{B\left[\kappa_{s0} + \frac{1}{2}(\tilde{v}'\kappa_{y0} + \tilde{u}'\kappa_{x0})\right]\right\}'$$
$$-T\left[\kappa_{s0} + \frac{1}{2}(\tilde{v}'\kappa_{y0} + \tilde{u}'\kappa_{x0})\right] + W(\phi' + \tilde{u}'\kappa_{x0} + \tilde{v}'\kappa_{y0} + \tilde{w}'\kappa_{s0})\kappa_{s0}$$
$$= Q_{es} + M_{ey}\kappa_{x0} - M_{ex}\kappa_{y0} \quad \text{or} \quad \mathrm{d}w = 0 \tag{68}$$

and

$$B\left[\kappa_{s0} + \frac{1}{2}(\tilde{u}'\kappa_{x0} + \tilde{v}'\kappa_{y0})\right] = 0 \quad \text{or} \quad \mathrm{d}w' = 0 \tag{69}$$

for axial compression in the direction of the axis os, and

$$W(\phi' + \tilde{u}'\kappa_{x0} + \tilde{v}'\kappa_{y0} + \tilde{w}'\kappa_{s0}) + B' - T = M_{es} \quad \text{or} \quad \mathrm{d}\phi = 0 \tag{70}$$

$$B = 0 \quad \text{or} \quad \mathrm{d}\phi' = 0 \tag{71}$$

for torsion about the axis os.

It is noted that the terms containing Wagner moment W appear in all the differential equilibrium equations and boundary conditions for bending deformations in the axes ox and oy, axial deformations in the axis os and torsional deformations about the axis os.

7. Applications

7.1. Beams curved in plan

The nonlinear differential equations of equilibrium can be applied directly to various problems and can also be used with a number of numerical methods such as the finite difference, finite element, finite strip, and Hermite methods. For example, they can be applied to the nonlinear analysis of beams curved in plan. For beams curved in plan without initial twist, $\kappa_{s0} = \kappa_{x0} = 0$. When the effect of the second order curvature κ_{y0}^2 is ignored, Eqs. (58)–(61) reduce to the nonlinear differential equilibrium equations of beams curved in plan that take the form

$$-N(1 + \tilde{w}')\kappa_{y0} - [N(u' + w\kappa_{y0})]' + [M_xS]'' - \frac{1}{2}M_xC\kappa_{y0}^2 + [M_yC]'' - \frac{1}{2}[Bv''']' + \frac{1}{2}[Bv']''' - \frac{1}{2}[Tv'']' - \frac{1}{2}[Tv']''$$
$$= q_x - m'_{ey} \tag{72}$$

for bending about the axis oy,

$$-[Nv']'-[M_xC]''+[M_yS]''+\frac{1}{2}M_yC\kappa_{y0}^2+(M_yC\tilde{v}'\kappa_{y0})'+\frac{1}{2}[B(u'''+w''\kappa_{y0})]'-[B\kappa_{y0}]''-\frac{1}{2}[B(u'+w\kappa_{y0})]'''$$
$$+\frac{1}{2}[T(u''+w'\kappa_{y0})]'+[T\kappa_{y0}]'+\frac{1}{2}[T(u'+w\kappa_{y0})]''-[W(\phi'+v'\kappa_{y0})\kappa_{y0}]'=q_y+m'_{ex} \tag{73}$$

for bending about the axis ox,

$$N[(u'+w\kappa_{y0})\kappa_{y0}]-N'-\frac{1}{2}[M_x\kappa_{y0}S]'-\frac{1}{2}[M_y\kappa_{y0}C]'+\frac{1}{2}Bv'''\kappa_{y0}-\frac{1}{2}[Bv'\kappa_{y0}]''+\frac{1}{2}Tv''\kappa_{y0}+\frac{1}{2}[Tv'\kappa_{y0}]'$$
$$=q_s+m_{ey}\kappa_{y0} \tag{74}$$

for axial compression in the direction of the axis os, and

$$M_x\left[\left(u''+\frac{1}{2}w'\kappa_{y0}\right)C+v''S+\kappa_{y0}C\right]-M_y\left[\left(u''+\frac{1}{2}w'\kappa_{y0}\right)S-v''C+\kappa_{y0}S\right]-B''+T'-\left[W(\phi'+v'\kappa_{y0})\right]'$$
$$=m_{es} \tag{75}$$

for torsion about the axis os. Eqs. (72) and (75) are consistent with those of Pi and Trahair (1997) and close to those of Nakai and Yoo (1988).

A steel I-section beam curved in plan (Fig. 4) with a constant radius that was considered by Fukumoto and Nishida (1981) has been analysed. The curved beam is simply supported at both ends (in plane $v(0)=v(S)=w(0)=0$ and out-of-plane $u(0)=u(S)=\phi(0)=\phi(S)=0$) and subjected to equal end moments.

The cross-section has an overall height $D=250$ mm, a flange width $B=100$ mm, a flange thickness $t_f=8$ mm and a web thickness $t_w=5.54$ mm. The material properties are Young's modulus $E=2.06\times10^5$ MPa, Poisson's ratio $v=0.3$, and the yield stress $\sigma_y=235$ MPa. A linear elastic stress–strain relationship was used in the analysis. The radius of the curved beam is $R=32.24$ m and its length is $S=2.579$ m, so that the include angle is $\Theta=4.583°$, and the ratio of the curved beam offset to its chord length is 1/100.

Variations of the central twist rotation ϕ and the dimensionless central in-plane and out-of-plane displacements v/D, u/B with the dimensionless end moments M/M_{yz} are shown in Fig. 5. Also shown in Fig. 5 are the numerical results of Fukumoto and Nishida (1981) obtained by the transfer matrix method. The results obtained from Eqs. (72) to (75) can be seen to be quite close to those of Fukumoto and Nishida.

7.2. Flexural–torsional buckling of circular arches

7.2.1. Doubly symmetric arches in uniform compression

The nonlinear differential equations of equilibrium given by Eqs. (58)–(61) can also be applied to the investigation of the flexural–torsional buckling analysis of circular arches. For a circular arch without

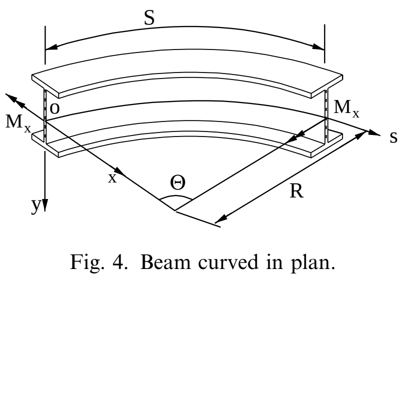

Fig. 4. Beam curved in plan.

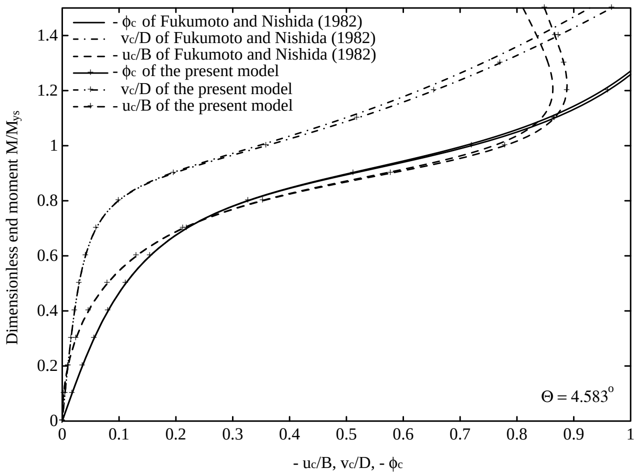

Fig. 5. Comparison with Fukumoto's results for beams curved in plan.

initial twist, the initial curvature κ_{y0} about the axis *oy* and the initial twist κ_{s0} are is equal to zero and the curvature κ_{x0} about the axis *ox* (i.e. in the plane *oys* as shown in Figs. 3 and 6) is negative, since it is in the positive direction of the axis *oy*. Hence, the magnitude *R* of the radius of the arch is given by

$$R = -\frac{1}{\kappa_{x0}} \tag{76}$$

When the arch is subjected to in-plane loads, it may suddenly bifurcate from its prebuckled equilibrium position $\{0, v, w, 0\}$ by deflecting laterally u_{b} and twisting ϕ_{b} to a new buckled equilibrium position $\{u_{\mathrm{b}}, v, w, \phi_{\mathrm{b}}\}$ under the constant conservative loads.

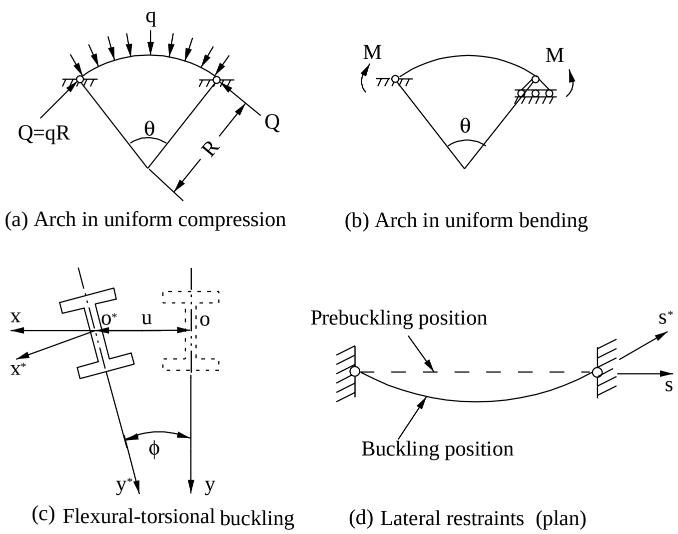

Fig. 6. Flexural–torsional buckling of arches in uniform compression or bending.

In buckling analysis, it is sufficient to assume the prebuckling behaviour of an arch is linear and so only the stress resultants corresponding to the prebuckling in-plane deformation exist. The stress resultants defined by Eqs. (44), (46) and (48) can then be linearised as

$$N = \int_A \sigma_{ss}\,\mathrm{d}A \tag{77}$$

$$M_x = \int_A \sigma_{ss} y\,\mathrm{d}A \tag{78}$$

and

$$W = \int_A \sigma_{ss}(x^2+y^2)\,\mathrm{d}A = EI_\mathrm{p}(w' + v\kappa_{x0}) = \frac{I_\mathrm{p}}{A}N = r_0^2 N \tag{79}$$

where A is the area of the cross-section, I_x is the second moment of area about the major principal axis of the cross-section, I_p is the polar moment of area of the cross-section and which are defined by

$$A = \int_A \mathrm{d}A, \quad I_x = \int_A y^2\,\mathrm{d}A, \quad I_\mathrm{p} = \int_A (x^2+y^2)\,\mathrm{d}A \tag{80}$$

Because the twist rotation ϕ_b of the cross-section is small during flexural–torsional buckling, it is sensible to assume that $\sin\phi_\mathrm{b} \approx \phi_\mathrm{b}$ and $\cos\phi_\mathrm{b} \approx 1$. The stress resultants corresponding to the buckling displacements u_b and ϕ_b can also obtained from Eqs. (45), (47) and (49) as

$$M_y = -\int_A \sigma_{ss} x\,\mathrm{d}A = EI_y(u_\mathrm{b}'' - \phi_\mathrm{b}\kappa_{x0}) = EI_y(u_\mathrm{b}'' + \phi_\mathrm{b}/R) \tag{81}$$

$$B = \int_A \sigma_{ss}\omega\,\mathrm{d}A = -EI_\mathrm{w}(\phi_\mathrm{b}'' + u_\mathrm{b}''\kappa_{x0}) = -EI_\mathrm{w}(\phi_\mathrm{b}'' - u_\mathrm{b}''/R) \tag{82}$$

and

$$T = \int_A \left[\left(x - \frac{\partial\omega}{\partial y}\right)\tau_{sy} - \left(y + \frac{\partial\omega}{\partial x}\right)\tau_{sx}\right]\mathrm{d}A = -GJ(\phi_\mathrm{b}' + u_\mathrm{b}'\kappa_{x0}) = -GJ(\phi_\mathrm{b}' - u_\mathrm{b}'/R) \tag{83}$$

A primary uniform prebuckling compression force $Q_s = -N$ in a circular arch is produced by a radial load $q_y = Q_z/R$ uniformly distributed around the arch (Fig. 6(a)). In this case, the in-plane bending moment $M_x = 0$. The differential equations of equilibrium for flexural–torsional buckling of an arch in uniform compression can also be obtained by substituting Eqs. (79)–(83) into Eqs. (58) and (61) and considering $M_x = 0$ as

$$\{Q_s u_\mathrm{b}'\}' - \{r_0^2 Q_s(\phi_\mathrm{b}' - u_\mathrm{b}'/R)/R\}' + \{EI_y(u_\mathrm{b}'' + \phi_\mathrm{b}/R)\}'' - \{EI_\mathrm{w}(\phi_\mathrm{b}'' - u_\mathrm{b}''/R)/R\}'' + \{GJ(\phi_\mathrm{b}' - u_\mathrm{b}'/R)/R\}' = 0 \tag{84}$$

$$\{r_0^2 Q_s(\phi_\mathrm{b}' - u_\mathrm{b}'/R)\}' + EI_y(u_\mathrm{b}'' + \phi_\mathrm{b}/R)/R + \{EI_\mathrm{w}(\phi_\mathrm{b}'' - u_\mathrm{b}''/R)\}'' - \{GJ(\phi_\mathrm{b}' - u_\mathrm{b}'/R)\}' = 0 \tag{85}$$

When the arch is pin-ended out-of-plane, its nth mode buckled shape (Fig. 6(a)) can be defined by

$$\frac{u_\mathrm{b}}{\delta} = \frac{\phi_\mathrm{b}}{\theta} = \sin\frac{n\pi s}{S} \tag{86}$$

where δ and θ are the maximum lateral displacement and twist angle and which satisfies the kinematical boundary conditions $u_\mathrm{b}(0) = u_\mathrm{b}(S) = 0$ and $\phi_\mathrm{b}(0) = \phi_\mathrm{b}(S) = 0$, and the static boundary conditions $EI_y u_\mathrm{b}''(0) = EI_y u_\mathrm{b}''(S) = 0$ and $EI_\mathrm{w}\phi_\mathrm{b}''(0) = EI_\mathrm{w}\phi_\mathrm{b}''(S) = 0$.

Substituting Eq. (86) into Eqs. (84) and (85) leads to the homogeneous algebraic equations

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \delta \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{87}$$

with

$$k_{11} = \left\{ 1 + a_n^2 b_n^2 - \left(1 + a_n^2 b_n^2 \frac{N_{yn}}{N_{sn}} \right) \frac{Q_s}{N_{yn}} \right\} N_{yn} \tag{88}$$

$$k_{12} = k_{21} = -\left\{ \frac{a_n}{b_n} + a_n b_n - \frac{N_{yn}}{N_{sn}} \frac{Q_s}{N_{yn}} \right\} M_{nys} \tag{89}$$

$$k_{22} = \left\{ 1 + \frac{a_n^2}{b_n^2} - \frac{N_{yn}}{N_{sn}} \frac{Q_s}{N_{yn}} \right\} r_0^2 N_{sn} \tag{90}$$

where

$$a_n = \frac{S}{n\pi R}, \quad b_n = \frac{n\pi M_{ysn}}{N_{yn} S} \tag{91}$$

$$M_{ysn} = r_0^2 N_{yn} N_{sn}, \qquad N_{yn} = \frac{EI_y (n\pi)^2}{S^2}, \qquad N_{sn} = \frac{1}{r_0^2} \left(GJ + \frac{EI_{\mathrm{w}} (n\pi)^2}{S^2} \right) \tag{92}$$

Solving Eq. (87) in conjunction with Eqs. (88) and (90) gives the buckling compressive load Q_s as

$$\frac{Q_s}{N_{yn}} = \frac{1}{2} \left(1 + \frac{a_n^2}{b_n^2} \right) \frac{N_{sn}}{N_{yn}} + \frac{1}{2} (1 - a_n^2)^2 - \sqrt{ \left\{ \frac{1}{2} \left(1 + \frac{a_n^2}{b_n^2} \right) \frac{N_{sn}}{N_{yn}} + \frac{1}{2} (1 - a_n^2)^2 \right\}^2 - (1 - a_n^2)^2 \frac{N_{sn}}{N_{yn}} } \tag{93}$$

which is the same as that of Pi and Bradford (2002), close to that obtained by Trahair and Bradford (1998), but differs from that obtained by Vlasov (1961). It has been pointed out by Trahair and Bradford (1998) and Pi and Bradford (2002) that Vlasov's result (1961) for the flexural–torsional buckling load of a pin-ended doubly symmetric arch in uniform compression is incorrect.

It is worth pointing out that the terms $-\{r_0^2 Q_s (\phi_{\mathrm{b}}' - u_{\mathrm{b}}'/R)/R\}'$ in Eq. (84) and $\{r_0^2 Q_s (\phi_{\mathrm{b}}' - u_{\mathrm{b}}'/R)\}'$ in Eq. (85) are those corresponding to the Wagner moment given by Eq. (79). The solution without these terms (Papangelis and Trahair, 1987) overestimates the flexural–torsional buckling load.

7.2.2. Doubly symmetric arches in uniform bending

An arch that is simply supported in its plane of loading subjected to two equal and opposite end moments is primarily in uniform bending (Fig. 6(b)). It is well known that in the classical flexural–torsional buckling analysis, the effects of prebuckling in-plane deformations on the flexural–torsional buckling are usually ignored (Timoshenko and Gere, 1961; Vlasov, 1961; Trahair and Bradford, 1998), and that the flexural–torsional buckling moment of beams under uniform bending without the effects of the prebuckling in-plane deformations are widely used in a number of steel structure design codes such as American Institute of Steel Construction (2000), British Standards Institution (1998), and Standards Australia (1998). Hence, the flexural–torsional buckling of arches under uniform bending without the effects of the prebuckling in-plane deformations is investigated in this section. In this case, substituting the linearised stress resultants M_y, B and T, given by Eqs. (81)–(83) into Eqs. (58) and (61) and considering $M_x = M$ and $N = 0$ leads to the following differential equations of equilibrium for flexural–torsional buckling of an arch in uniform bending

$$\{M_x\phi_b\}'' - \{M_x u_b'/R\}' + \{EI_y(u_b'' + \phi_b/R)\}'' - \{EI_w(\phi_b'' - u_b''/R)/R\}'' + \{GJ(\phi_b' - u_b'/R)/R\}' = 0 \tag{94}$$

$$M_x(u_b'' + \phi_b/R) + EI_y(u_b'' + \phi_b/R)/R + \{EI_w(\phi_b'' - u_b''/R)\}'' - \{GJ(\phi_b' - u_b'/R)\}' = 0 \tag{95}$$

When the arch is pin-ended out-of-plane, its nth mode buckling shape can also be defined by Eq. (86). Substituting Eq. (86) into Eqs. (94) and (95) leads to Eq. (87), but with

$$k_{11} = \left(1 + a_n^2 b_n^2 + a_n b_n \frac{M_x}{M_{ysn}}\right) N_{yn} \tag{96}$$

$$k_{12} = k_{21} = -\left(\frac{a_n}{b_n} + a_n b_n + \frac{M_x}{M_{ysn}}\right) M_{ysn} \tag{97}$$

$$k_{22} = \left(1 + \frac{a_n^2}{b_n^2} + \frac{a_n}{b_n}\frac{M_x}{M_{ysn}}\right) r_0^2 N_{sn} \tag{98}$$

As for the case of arches under uniform compression, the buckling moment M_x is obtained as

$$\frac{M_x}{M_{ysn}} = -\frac{a_n}{2}\left(b_n + \frac{1}{b_n}\right) \pm \sqrt{\frac{a_n^2}{4}\left(b_n + \frac{1}{b_n}\right)^2 + (1 - a_n^2)} \tag{99}$$

which is the same as those obtained by Timoshenko and Gere (1961) and Vlasov (1961).

When the effects of prebuckling in-plane deformations on the flexural–torsional buckling of arches under uniform bending are considered, the terms containing the in-plane deformations in Eqs. (58) and (61) need to be considered. The in-plane curvature $\tilde{v}''$ produced by prebuckling in-plane deformations can be obtained by linearising the bending moment M_x due to prebuckling in-plane deformations given by Eq. (78) as

$$M_x = \int_A \sigma_{ss} y \, dA = \int_A E\epsilon_{ss} y \, dA = -\tilde{v}'' EI_x \tag{100}$$

from which

$$\tilde{v}'' = -\frac{M_x}{EI_x} \tag{101}$$

In this case, the coefficients k_{11}, k_{12}, k_{21}, and k_{22} in the homogeneous algebraic equations (87) can be obtained as

$$k_{11} = \left[1 + a_n^2 b_n^2 + a_n b_n (1 - b_n^2) \frac{M_x}{M_{ysn}}\right] N_{yn} \tag{102}$$

$$k_{12} = -\left[\frac{a_n}{b_n} + a_n b_n + (1 - b_n^2)\frac{M_x}{M_{ysn}}\right] M_{ysn} \tag{103}$$

$$k_{21} = -\left[\frac{a_n}{b_n} + a_n b_n + \left(1 - \frac{I_y}{I_x}\right)\frac{M_x}{M_{ysn}}\right] M_{ysn} \tag{104}$$

$$k_{22} = \left[1 + \frac{a_n^2}{b_n^2} + \frac{a_n}{b_n}\left(1 - \frac{I_y}{I_x}\right)\frac{M_x}{M_{ysn}}\right] r_0^2 N_{sn} \tag{105}$$

In the same way as for the case of arches under uniform bending without effects of prebuckling in-plane deformations, the buckling moment M_x is obtained as

$$\frac{M_x}{M_{ysn}} = \frac{-D \pm \sqrt{D^2 + 4(1 - I_y/I_x)(1 - b_n^2)(1 - a_n^2)}}{2(1 - I_y/I_x)(1 - b_n^2)} \tag{106}$$

with $D = a_n b_n(1 - I_y/I_x) + a_n/b_n(1 - b_n^2)$, which is consistent with those obtained by Vacharajittiphan and Trahair (1975), Hirashima et al. (1979) and Pi et al. (1995).

7.3. Nonlinear behaviour of members doubly-curved in space

The nonlinear elastic behaviour of a pin-ended member curved in space that is subjected to a central vertical load has been analysed using a finite spatially-curved beam element model that has been developed by authors elsewhere (Pi et al., 2003) on the basis of the nonlinear differential equations of equilibrium given by Eqs. (58)–(61). The dimensions of the cross-section and material properties of the member are the same as those of the beam given in Section 7.1. The length of the member is $S = 2\,\text{m}$. The centroidal axis of the member is a space curve and its initial curvatures about both the principal axes ox and oy are $\kappa_{x0} = 0.43633$ and $\kappa_{y0} = 0.008727$. For comparison, a pin-ended arch, a pin-ended beam curved in plan, and a simply supported beam curved in plan with the same cross-section and material properties, the same length and the same boundary and load conditions were also investigated. The curvature of the beam curved in plan is $\kappa_{y0} = 0.008727$ which corresponds to an included angle $\Theta = 1°$ while the curvature of the arch is $\kappa_{x0} = 0.43633$ which corresponds to an included angle $\Theta = 50°$. In the analysis, to trigger the three dimensional response of the arch, very small initial lateral load of 0.001 kN was conventionally applied to the crown of the arch.

Variations of the absolute value of the central twist rotation $|\phi|$with the central vertical load are shown in Fig. 7. It can be seen that the behaviour of the member curved in space is very different from those of the

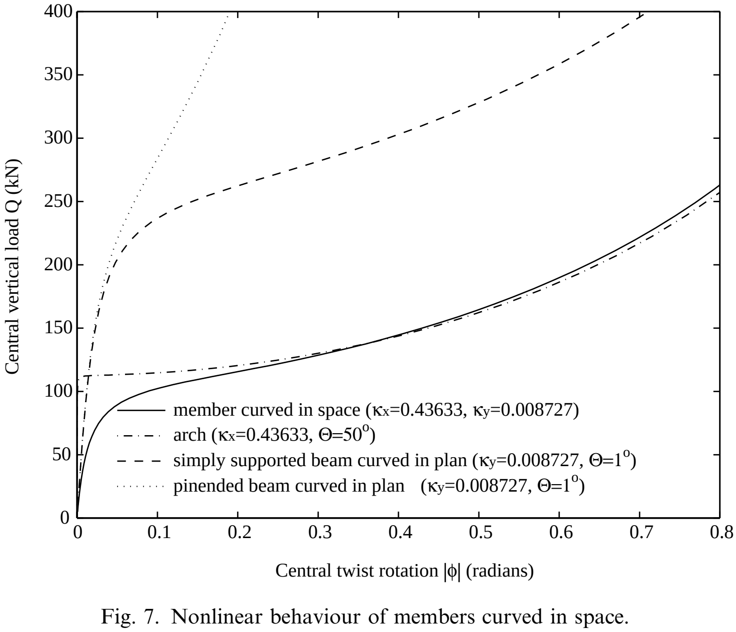

Fig. 7. Nonlinear behaviour of members curved in space.

arch and beams curved in plan. There is no twist rotation for the arch until an identifiable buckling load is reached while the twist rotation is a primary deformation for the beam curved in plan. Because the included angle $\Theta = 1^\circ$ is small, the behaviour of the simply supported beam curved in plan is similar to a simply supported straight beam with initial lateral imperfections. The behaviour of the pin-ended beam curved in plan is similar to that of its simply supported counterpart when the deformations are small. However, its response is much stiffer when the deformations are large because the induced tension and Wagner moment in the pin-ended beam curved in plan increase rapidly with an increase of the large deformations. The twist rotation is also a primary deformation for the member curved in space, and it can be seen that it becomes nonlinear very early and the nonlinear behaviour becomes significant as the deformations further increase.

8. Concluding remarks

A consistent nonlinear theory for the analysis of members curved in space with warping has been presented in this paper. The merits of the "geometrically exact beam theory" and the weak form formulation have been combined in the development. A rotation matrix of the special rotation group that satisfies orthogonality conditions was used in the derivations of deformed curvatures, position vectors, and nonlinear strains. Nonlinear differential equations of equilibrium for members curved in space were then established based on the principle of virtual work. It has been found that to predict the nonlinear behaviour of members curved in space, the bimoment and Wagner moment stress resultants and their conjugate finite strains have to be included in the nonlinear differential equations of equilibrium.

The nonlinear differential equations of equilibrium can be applied directly to various problems and can also be used with a number of numerical methods such as the finite difference, finite element, finite strip, and Hermite methods. They can be used for large deformation analysis of curved members that are subjected to axial compression, biaxial bending, uniform and nonuniform torsion and their combined actions. Applications of the nonlinear differential equations of equilibrium for the analyses of the nonlinear behaviour of beams curved in plan, the flexural–torsional buckling analysis of arches and the nonlinear behaviour of members curved in space have been demonstrated, and the results for curved beams and for arches were shown to agree with other studies. These comparisons with existing analytical and numerical results indicate that the nonlinear differential equations of equilibrium for members curved in space can provide reliable solutions for a number of problems.

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